Journal of Applied Mechanics and Technical Physics, Vol. 41, No. 6, 2000

ESTIMATE OF THE LATENT FREE ENERGY AND DAMAGE AT THE TIP OF AN OPENING-MODE CRACK

Yu. N. Radaev

UDC 539.3

A method of estimating the latent elastic energy associated with the microinhomogeneity of the stress and plastic-strain fields inside the plastic zone localized near the tip of an opening-mode crack (Dugdale zone) under conditions of plane stresses is proposed. The microinhomogeneity of plastic flow upon small strain hardening is taken into account only in the form of considerable distortion of the geometry of the free surfaces of the plastic zone. The damage that developes because of release of the latent free energy is estimated depending on the magnitude of the crack opening.

In studying the elastoplastic fracture of metals, it is necessary to estimate the latent elastic energy accumulated in the zones of localized plastic strain, which usually occur near stress concentrators. Precisely the latent free energy (according to the traditional thermodynamic classification) is expended substantially in the development of damages under appropriate conditions (sign-alternating cyclic loading, the action of various embrittlement factors, in particular, an aggressive ambient medium, radiation, etc.).

In the present paper, a method of estimating the latent free energy that is based on a comparison of the geometries of the ideal plastic flow and the plastic flow distorted by a microinhomogeneous stress field is considered. For metals with small strain hardening (mild steel and some aluminum alloys), the stress microinhomogeneity is manifested as the surface instability in the plastic-flow region; therefore, the distortion of the geometry of the free boundary and the slip field is the determining factor. This method gives an upper estimate of the latent energy associated with a strongly inhomogeneous stress field in the zone where the localization of plastic strains is very pronounced, provided it is possible to calculate the change in the normal stress averaged over the cross section of the body in the localization zone which is caused by distortion of the free boundary.

In this paper, we estimate the latent free energy and damage in the vicinity of the tip of an opening-mode crack in steel plates under plane stresses after the loading–unloading cycle. In this case, the highly localized plastic flow is modeled with allowance for the complex geometry of the slip field in the normal cross sections of the plate which are perpendicular to the crack line. Using this estimate, we determine a magnitude of crack opening at which uncontrollable cracking occurs in the neighborhood of the crack tip.

1. Formulation of the Problem and Basic Equations. We consider a thin metal plate with an internal cut $|x_1| \leqslant l$ subjected to the tensile stress σ_{22}^{∞} (Fig. 1). It is assumed that the crack length $2l$ is much smaller than the dimensions of the plate and much larger than the plate thickness $2a$. We denote the tensile yield point by Y.

It is known [1, 2] that, for mild steels, the plastic zones that occur in the vicinity of a crack tip are extremely thin bands continuing the crack. Based on the assumptions given below, Dugdale [1] proposed a crack model that corresponds to a similar, strongly localized yielding near the crack tip:

Samara State University, Samara 443011. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 41, No. 6, pp. 106–117, November–December, 2000. Original article submitted June 15, 1998; revision submitted November 17, 1999.

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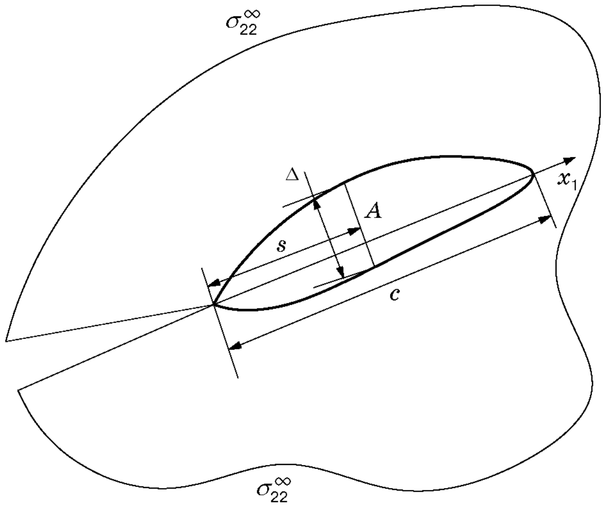

Fig. 1. Geometry of the localized plastic zone near the tip of an opening-mode crack under plane stresses.

1. The elements of the plastic zone are subjected only to the normal tensile stress equal to the tensile yield point Y.

2. The thickness of the plastic zone is much smaller than its length, so that the inner boundary can be regarded as a very flattened ellipse whose major semiaxis is equal to $l + c$, where c is the length of the plastic zone.

3. The length of the plastic zone c is such that this zone involves the normal-stress singularity.

The specific features of plastic flow near the crack are as follows: the plastic-strain field is very inhomogeneous, the strain hardening is almost absent (which substantiates the first assumption mentioned above), and the strong surface instability occurs, which results in considerable qualitative distortion of the plate geometry within the plastic region. Within the limits of the localized plastic zone, the thickness-averaged normal stresses can exceed the yield point Y. This is also typical of the localized axisymmetric flow (neck) [3]. In the absence of hardening, the normal stress averaged over the cross section of the localized plastic zone increases only due to distortion of the geometry of the free boundary; therefore, it is necessary to analyze in greater detail the geometry of plastic flow in the planes normal to the crack line and the plate plane. Moreover, in contrast to Dugdale's model, one should not ignore the extent of the plastic zone in the direction normal to the crack edges. Nevertheless, since the Dugdale assumptions allow one to calculate the linear dimension of the plastic zone and the displacement distribution near the crack tip with reasonable accuracy, the corresponding formulas can be used to estimate the length of the plastic zone and the magnitude of the crack opening.

We assume that within the plastic zone localized at the crack tip, the stresses and strains are decomposed into a regular (average) part and fluctuations caused by the distribution microinhomogeneity: $\sigma_{ij} = \bar{\sigma}_{ij} + \sigma_{ij}^{\mu}$ and $\varepsilon_{ij} = \bar{\varepsilon}_{ij} + \varepsilon_{ij}^{\mu}$, where averaging is performed over the elementary volume τ whose characteristic linear dimension is comparable to the typical scale of the plastic-flow inhomogeneity:

$$\bar{\sigma}_{ij} = \frac{1}{|\tau|} \iiint\limits_{\tau} \sigma_{ij}\, d\tau, \qquad \bar{\varepsilon}_{ij} = \frac{1}{|\tau|} \iiint\limits_{\tau} \varepsilon_{ij}\, d\tau.$$

The average stress and strain fluctuations vanish:

$$\iiint\limits_{\tau} \sigma_{ij}^{\mu}\, d\tau = 0, \qquad \iiint\limits_{\tau} \varepsilon_{ij}^{\mu}\, d\tau = 0.$$

Given that the distribution inhomogeneity is pronounced only in the plastic-strain localization zone, we can assume that the fluctuations vanish outside the plastic zone. Furthermore, if the stresses and the plastic strains are assumed to be continuous in passing from the plastic to the elastic zone, the stress and strain fluctuations also vanish at the boundary of the plastic zone.

From the energy viewpoint, the inhomogeneous plastic flow is characterized by the fact that a part of its free energy corresponding to the microstresses σ_{ij}^{μ} exists in the latent form and can be released under certain conditions, for example, under sign-alternating cyclic loading.

To estimate the latent free energy in the case of isothermal microinhomogeneous plastic flow, we consider the expression for the work of stresses done in small strain of a volume element $d\varepsilon_{ij}$:

$$\frac{1}{|\tau|}\iiint\limits_{\tau} \sigma_{ij}\, d\varepsilon_{ij}\, d\tau = \bar{\sigma}_{ij}\, d\bar{\varepsilon}_{ij} + \frac{1}{|\tau|}\iiint\limits_{\tau} \sigma_{ij}^{\mu}\, d\varepsilon_{ij}^{\mu}\, d\tau. \tag{1.1}$$

We decompose the total and average strains into the elastic and plastic components:

$$\varepsilon_{ij} = \varepsilon_{ij}^{E} + \varepsilon_{ij}^{P}, \qquad \bar{\varepsilon}_{ij} = \overline{\varepsilon_{ij}^{E}} + \overline{\varepsilon_{ij}^{P}}.$$

These relations imply a similar decomposition of the fluctuation of the strain tensor: $\varepsilon_{ij}^{\mu} = \varepsilon_{ij}^{\mu E} + \varepsilon_{ij}^{\mu P}$.

For the microscopic stress tensor σ_{ij}^{μ}, the role of elastic reversible strains is played by the plastic microstrains $\varepsilon_{ij}^{\mu P}$, since the stresses σ_{ij}^{μ} occur and vanish together with the strains $\varepsilon_{ij}^{\mu P}$. We assume that, for any cycle closed relative to plastic microstrains, the total work of the microstresses σ_{ij}^{μ} done in the microplastic strains $\varepsilon_{ij}^{\mu P}$ vanish:

$$\oint \sigma_{ij}^{\mu}\, d\varepsilon_{ij}^{\mu P} = 0. \tag{1.2}$$

This assumption is used in many plasticity theories that take into account the plastic-flow microinhomogeneity.

Consequently, the latent free energy ψ^{μ} (per unit volume) can be determined as the potential of microstresses

$$\sigma_{ij}^{\mu} = \frac{\partial \psi^{\mu}}{\partial \varepsilon_{ij}^{\mu P}},$$

which exists if condition (1.2) holds. In this case, Eq. (1.1) becomes

$$d\,\overline{\psi^{\mu}} = \frac{1}{|\tau|}\iiint\limits_{\tau} \sigma_{ij}\, d\varepsilon_{ij}\, d\tau - \frac{1}{|\tau|}\iiint\limits_{\tau} \sigma_{ij}^{\mu}\, d\varepsilon_{ij}^{\mu E}\, d\tau - \bar{\sigma}_{ij}\, d\bar{\varepsilon}_{ij}.$$

Assuming that the fluctuations of the elastic strain component are negligible ($\varepsilon_{ij}^{\mu E} = 0$ and $\varepsilon_{ij}^{E} = \overline{\varepsilon_{ij}^{E}}$), we obtain the equation

$$d\,\overline{\psi^{\mu}} = \frac{1}{|\tau|}\iiint\limits_{\tau} \sigma_{ij}\, d\varepsilon_{ij}^{P}\, d\tau - \bar{\sigma}_{ij}\, d\,\overline{\varepsilon_{ij}^{P}}.$$

Thus, the change in the latent free energy averaged over the characteristic volume is determined by the difference between the irreversible work done in the plastic deformation distorted by the internal microinhomogeneity and that done in the ideal plastic deformation. The last equation can also be written in the form

$$d\,\overline{\psi^{\mu}} = \overline{\sigma_{ij}\, d\varepsilon_{ij}^{P}} - \bar{\sigma}_{ij}\, d\,\overline{\varepsilon_{ij}^{P}} \tag{1.3}$$

and can be considered as a definition of the quantity $\overline{\psi^{\mu}}$, i.e., the density (per unit volume) of the latent free energy of microstresses.

Using the principle of energy equivalence, within the framework of the continual model, we introduce the microinhomogeneous extra-stress tensor Σ_{ij} in such a manner that the equality

$$\overline{\sigma_{ij}^{\mu}\, d\varepsilon_{ij}^{\mu P}} = \Sigma_{ij}\, d\,\overline{\varepsilon_{ij}^{P}}$$

holds. We note that from the energy viewpoint, this tensor determines the microfluctuations.

This equality does not determine the extra-stress tensor Σ_{ij} uniquely, but it is sufficient to assume hereinafter that the unique determination exists. It is noteworthy that, in accordance with the principle of energy equivalence, the stress and plastic-strain fluctuations depend on the current plastic strains averaged over the characteristic volume.

Equation (1.3) can be written in terms of the nonfluctuating tensor Σ_{ij} as follows:

$$d\,\overline{\psi^\mu} = \Sigma_{ij}\, d\,\overline{\varepsilon^P_{ij}}.$$

Here $\overline{\psi^\mu}$ is the potential of the extra-stress tensor Σ_{ij} which characterizes the microinhomogeneity of plastic flow in isothermal processes.

One can consider the specific form of the potential of state $\overline{\psi^\mu}$ and obtain a governing equation relating the extra-stress tensor Σ_{ij} to the plastic strains averaged over the characteristic volume. Within the framework of the continual model, this leads to the classical Ishlinskii–Prager equations or complex versions of them. The distinguishing feature of the approach proposed is that, for low-alloy steels, kinematic hardening is small; therefore, the effect of the plastic-flow microinhomogeneity can be taken into account only in the form of the surface instability of the plastic zone, i.e., the significant distortion of the free surface compared to its geometry for the ideal plastic flow.

As far as the author is aware, Eq. (1.3) was not considered previously. As is shown below, it can be used to estimate the damage near the crack tip in the plate. To this end, it is necessary to determine the total (for all elements of the localized plastic zone) increment of the latent free energy and calculate the total latent free energy by integrating over the entire loading process.

Let C be a certain load or geometrical parameter that characterizes the loading process and increases monotonically during loading. In the case of an opening-mode crack in a plate, if the normal tensile stress σ_{22}^∞ increases and reaches the maximum $\hat\sigma_{22}^\infty$, one can choose the increasing dimension of the plastic zone c as the parameter C (Fig. 1).

Integrating Eq. (1.3) over all elements of the current plastic zone V' and integrating the resulting relation over the entire process, we obtain the equation

$$\int_0^{\hat c} dc \iiint_{V'} dx_1\, dx_2\, dx_3\, \frac{d\,\overline{\psi^\mu}}{dc} = J - I, \tag{1.4}$$

where

$$J = \int_0^{\hat c} dc \iiint_{V'} dx_1\, dx_2\, dx_3\, \frac{\overline{\sigma_{ij} d\varepsilon^P_{ij}}}{dc}; \tag{1.5}$$

$$I = \int_0^{\hat c} dc \iiint_{V'} dx_1\, dx_2\, dx_3\, \bar\sigma_{ij}\, \frac{d\,\overline{\varepsilon^P_{ij}}}{dc}. \tag{1.6}$$

We transform the integral on the left side of Eq. (1.4). To this end, we write the integral over the plastic zone as a double integral [the internal integral is taken over the plane region A which is the intersection of the plastic zone and the plane normal to the plate and the crack line and located at the distance s from the crack tip; the external integral is taken over the crack line (Fig. 1)]:

$$\iiint_{V'} dx_1\, dx_2\, dx_3\, \frac{d\,\overline{\psi^\mu}}{dc} = \int_0^{\hat c} ds \iint_A dA\, \frac{d\,\overline{\psi^\mu}}{dc}.$$

We denote the relative displacement of the points located on the opposite sides of the plastic zone in the direction normal to the crack by $\delta(s,c)$. Assuming that s decreases with increase in s, we introduce the new variable $s = s(\delta, c)$:

$$\int_0^{\hat c} ds \iint_A dA\, \frac{d\,\overline{\psi^\mu}}{dc} = \int_0^{\omega(c)} \Big(-\frac{\partial s}{\partial \delta}\Big)_c\, d\delta \iint_A dA \frac{d\,\overline{\psi^\mu}}{dc}. \tag{1.7}$$

In this equation, the condition $\delta(c,c)=0$ is taken into account and the notation $\omega(c)=\delta(0,c)$ is used. In the first approximation, $\omega(c)$ can be considered as a magnitude of crack opening which depends on the length of the plastic zone.

It is convenient to use the parameter ω instead of c. Changing the integration variable along the process and using (1.7), we obtain

$$\int_0^{\hat{c}} dc \iiint_{V'} dx_1\,dx_2\,dx_3\,\frac{d\,\overline{\psi^\mu}}{dc} = \int_0^{\hat{\omega}} d\omega \int_0^{\omega} \Big(-\frac{\partial s}{\partial \delta}\Big)_{\omega} d\delta \iint_A dA\,\frac{d\overline{\psi^\mu}}{d\omega}.$$

The integral over δ is approximately calculated with the use of the mean-value theorem:

$$\int_0^{\hat{c}} dc \iiint_{V'} dx_1\,dx_2\,dx_3\,\frac{d\,\overline{\psi^\mu}}{dc} = \int_0^{\hat{\omega}} \omega\Big(-\frac{\partial s}{\partial \delta}\Big)_{\omega,\bar{\delta}} \iint_{\bar{A}} \frac{d}{d\omega}\,(\overline{\psi^\mu})_{\bar{\delta}}\,dA\,d\omega,$$

where $\bar{A}$ is a certain characteristic cross section of the plastic zone.

We assume that the latent free energy $\overline{\psi^\mu}$ varies slightly within $\bar{A}$ (or, alternatively, this variation can be ignored in comparison with variation in $\overline{\psi^\mu}$ associated with an increase in ω) and the area $\bar{A}$ changes slightly during loading. As a result, we obtain

$$\int_0^{\hat{c}} dc \iiint_{V'} dx_1\,dx_2\,dx_3\,\frac{d\,\overline{\psi^\mu}}{dc} = \iint_{\bar{A}} dA \int_0^{\hat{\omega}} \omega\Big(-\frac{\partial s}{\partial \delta}\Big)_{\omega,\bar{\delta}} \frac{d}{d\omega}\,(\,\overline{\psi^\mu}\,)_{\bar{\delta}}\,d\omega.$$

To calculate the internal integral over $d\omega$, we approximate δ by a function of the form $\delta(\omega(c)(c-s))$ and integrate by parts. Finally, we have

$$\int_0^{\hat{c}} dc \iiint_{V'} dx_1\,dx_2\,dx_3\,\frac{d\,\overline{\psi^\mu}}{dc} = \hat{\omega}\Big(-\frac{\partial s}{\partial \delta}\Big)_{\hat{\omega},\bar{\delta}} \iint_{\bar{A}} (\,\overline{\psi^\mu}\,)_{\hat{\omega},\bar{s}}\,dA. \tag{1.8}$$

The integral J can be evaluated according to the following procedure. We reduce the volume integral in (1.5) to a double integral as shown above. Then, using the mean-value theorem, we calculate the integral over s and replace the integration variable c by the variable $\bar{\delta}$, which is determined from equation $c=c(\bar{s},\bar{\delta})$. As a result, we obtain

$$J = \int_0^{\hat{\delta}(\bar{s})} c(\bar{s},\bar{\delta})\,d\bar{\delta} \iint_{\bar{A}} dA\,\frac{\overline{\sigma_{ij}\,d\varepsilon^P_{ij}}}{d\bar{\delta}}.$$

Replacing $c(\bar{s},\bar{\delta})$ by $\hat{c}$, which is the maximum possible value for the entire loading process, we finally obtain

$$J = \hat{c}\int_0^{\hat{\delta}(\bar{s})} d\bar{\delta} \iint_{\bar{A}} dA\,\frac{\overline{\sigma_{ij}\,d\varepsilon^P_{ij}}}{d\bar{\delta}} \tag{1.9}$$

and a similar formula for the integral (1.6):

$$I = \hat{c}\int_0^{\hat{\delta}(\bar{s})} d\bar{\delta} \iint_{\bar{A}} dA\,\bar{\sigma}_{ij}\,\frac{d\,\overline{\varepsilon^P_{ij}}}{d\bar{\delta}}. \tag{1.10}$$

Using formulas (1.8)–(1.10), we write Eq. (1.4) in the form

$$\hat{\omega}\Big(-\frac{\partial s}{\partial \delta}\Big)_{\hat{\omega},\bar{\delta}} \iint_{\bar{A}} (\,\overline{\psi^\mu}\,)_{\hat{\omega},\bar{s}}\,dA = \hat{c}\int_0^{\hat{\delta}(\bar{s})} d\bar{\delta} \iint_{\bar{A}} dA\,\frac{\overline{\sigma_{ij}\,d\varepsilon^P_{ij}}}{d\bar{\delta}} - \hat{c}\int_0^{\hat{\delta}(\bar{s})} d\bar{\delta} \iint_{\bar{A}} dA\,\bar{\sigma}_{ij}\,\frac{d\,\overline{\varepsilon^P_{ij}}}{d\bar{\delta}}. \tag{1.11}$$

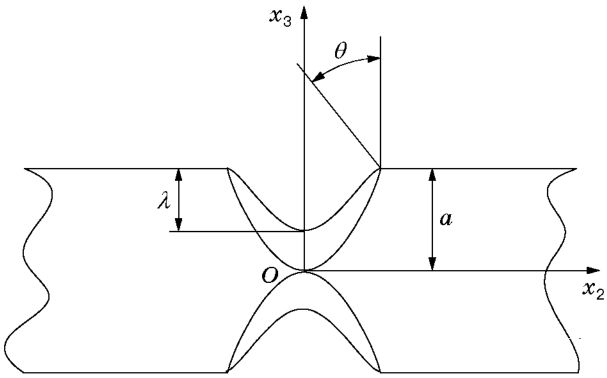

Fig. 2

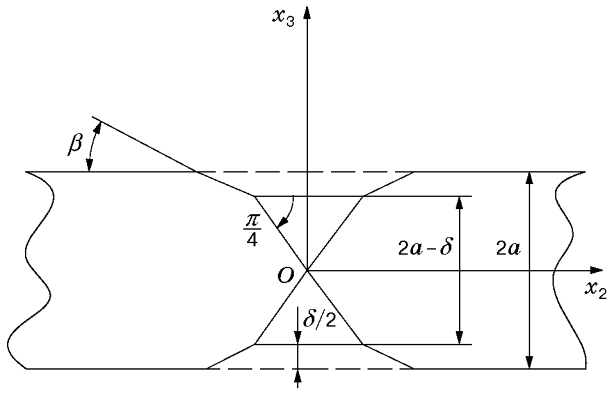

Fig. 3

Fig. 2. Cross section of the localized microinhomogeneous plastic zone.

Fig. 3. Geometry of the ideal plastic flow in the cross section of the plastic zone.

Below, we use Eq. (1.11) to estimate the latent free energy localized in the vicinity of the tip of an opening-mode crack under plane stress.

We also note that with allowance for the formula

$$\Big(-\frac{\partial s}{\partial \delta}\Big)_{\hat{\omega},\bar{\delta}} = \Big\{\Big(-\frac{\partial \delta}{\partial s}\Big)_{\hat{\omega},\bar{s}}\Big\}^{-1} = \Big\{\frac{1}{\hat{c}}\int\limits_0^{\hat{c}}\Big(-\frac{\partial \delta}{\partial s}\Big)_{c=\hat{c}}\,ds\Big\}^{-1} = \frac{\hat{c}}{\hat{\omega}}, \tag{1.12}$$

Eq. (1.11) becomes

$$\iint\limits_{\bar{A}}(\overline{\psi^{\mu}})_{\hat{\omega},\bar{s}}\,dA = \int\limits_0^{\hat{\delta}(\bar{s})} d\bar{\delta}\iint\limits_{\bar{A}} dA\,\frac{\overline{\sigma_{ij}\,d\varepsilon^P_{ij}}}{d\bar{\delta}} - \int\limits_0^{\hat{\delta}(\bar{s})} d\bar{\delta}\iint\limits_{\bar{A}} dA\,\bar{\sigma}_{ij}\,\frac{d\,\overline{\varepsilon^P_{ij}}}{d\bar{\delta}}.$$

2. Concentration of the Free Energy of Microstresses in the Middle Section of a Localized Plastic Zone. The values of each integral on the right side of Eq. (1.4) can be estimated if the characteristic linear dimension of the plastic zone is assumed to be larger than the plate thickness and, hence, the states close to plane strain occur in the cross sections of the plastic zone.

This assumption is supported by experimental data [4]: except for the extremely small zone at the crack tip, the slip system possesses all the characteristic features of plane plastic flow with the slip lines intersected at right angles. Therefore, to approximate the stresses, one can use the statically admissible stress fields constructed in accordance with the scheme of plane plastic flow.

Moreover, we assume that the magnitude of crack opening is much smaller than the plate thickness. Since the strain hardening is insignificant, the yield point Y remains unchanged under loading. Thus, the microinhomogeneity of stresses and strains is manifested only in the form of yield instability, and it is taken into account as considerable distortion of the plate and an increase in the average stress in the thinnest cross section of the plastic zone. Figure 2 shows the geometry of the plate cross section and the schematic form of the plastic zone.

We estimate the integral I with the use of the Onate–Prager scheme of the ideal rigid-plastic flow shown in Fig. 3 [$\beta = \arctan(1/2)$]. The flow geometry is quite simple and is determined by the incompressibility condition.

Denoting the total tensile force per unit length of the plastic zone in the direction of x_2 by F_δ,

$$F_\delta = 2\int\limits_0^{a-\delta/2}\sigma_{22}\,dx_3,$$

where a is half of the plate thickness, and taking into account that

$$2\int\limits_0^{\hat{\delta}/2} F_\delta d\,\frac{\delta}{2} = 4Y\Big(a\xi - \frac{\xi^2}{2}\Big)\Big|_{\xi=0}^{\hat{\delta}/2},$$

we obtain

$$I = 2\hat{c}\int\limits_0^{\hat{\delta}(\bar{s})/2} F_{\bar{\delta}} d\,\frac{\bar{\delta}}{2} = 4Y\hat{c}\Big(a\xi - \frac{\xi^2}{2}\Big)\Big|_{\xi=0}^{\hat{\delta}(\bar{s})/2}.$$

The corresponding estimates for the unstable plastic flow accompanied by distortion of the free-boundary geometry (see Fig. 2) are more complicated, since the form of the free boundary is not known in advance. In the plane-strain approximation, the stress state can be studied by the hodograph method [5]. Calculating the total tensile force F_δ in the rigid-plastic approximation, we obtain [5]

$$F_\delta = 2Y\Big\{a - \lambda(\delta) - \int\limits_0^{\theta(\delta)} æ(v,\delta)\,dv\Big\},$$

where the equations of the variable free surface are written in the parametric form $x_2 = \mu_2(u,\delta)$ and $x_3 = \mu_3(u,\delta)$, where the angle between the tangent to the free boundary and the horizontal axis is used as a parameter, $\lambda(\delta)$ is the maximum depth of thinning of the plate, and $\theta(\delta)$ is the angle between the normal to the free boundary and the vertical that intersects the rigid-plastic boundary (see Fig. 2),

$$æ(v,\delta) = \Lambda(v,\delta) + \frac{\partial\Lambda(v,\delta)}{\partial v} - (1+v)\int\limits_0^v \Lambda(t,\delta)\,\frac{I_1(\sqrt{t^2-v^2})}{\sqrt{t^2-v^2}}\,dt - v\int\limits_0^v \Lambda(t,\delta)\,\frac{\partial}{\partial v}\,\frac{I_1(\sqrt{t^2-v^2})}{\sqrt{t^2-v^2}}\,dt,$$

$$\Lambda(u,\delta) = \mu_2(u,\delta)\sin u - \mu_3(u,\delta)\cos u.$$

Here I_1 is the first-order Bessel function of imaginary argument. It is noteworthy that the expansion

$$\frac{I_1(\sqrt{t^2-v^2})}{\sqrt{t^2-v^2}} = \sum_{n=0}^{\infty}\frac{(v^2-t^2)^n}{2^{2n+1}n!(n+1)!}$$

is valid.

Since, as earlier, the integral in question is the total work of the normal stress averaged over the thinnest cross section of the plastic zone, we have

$$J = 2\hat{c}\int\limits_0^{\hat{\delta}(\bar{s})/2} F_{\bar{\delta}}\,d\,\frac{\bar{\delta}}{2}.$$

Hence, we obtain the estimate

$$J = 4Y\hat{c}\int\limits_0^{\hat{\delta}(\bar{s})/2}\Big\{a - \lambda(\bar{\delta}) - \int\limits_0^{\theta(\bar{\delta})} æ(v,\bar{\delta})\,dv\Big\}d\,\frac{\bar{\delta}}{2}.$$

As a result, Eqs. (1.11) and (1.12) can be combined to give

$$\iint\limits_{\bar{A}} (\,\overline{\psi^\mu}\,)_{\hat{\omega},\bar{s}}\,dx_2\,dx_3 = -2Y\xi^2\Big|_{\xi=0}^{\hat{\delta}(\bar{s})/2} - 4Y\int\limits_0^{\hat{\delta}(\bar{s})/2}\lambda(\bar{\delta})\,d\,\frac{\bar{\delta}}{2} + 4Y\int\limits_0^{\hat{\delta}(\bar{s})/2}\Big\{\int\limits_0^{\theta(\bar{\delta})}(-æ(v,\bar{\delta}))\,dv\Big\}d\,\frac{\bar{\delta}}{2}.$$

This equation allows one to obtain certain simple estimates of the latent free energy. It should be noted that the first term on the right side of the last equation is of second order in the magnitude of crack opening,

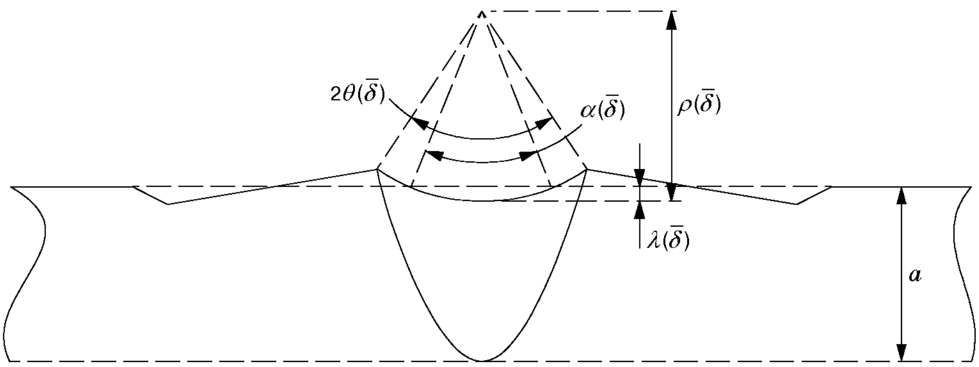

Fig. 4. The simplest model of the localized plastic zone with a curvilinear free surface.

and the second term is of the order of the product of the plate thinning and the magnitude of crack opening. Introducing the generalized force

$$\mathcal{F}_{\bar{\delta}} = 2\,\frac{\partial}{\partial\bar{\delta}}\,\frac{1}{a}\iint\limits_{\bar{A}} (\,\overline{\psi^{\mu}}\,)_{\bar{\delta},\bar{s}}\,dx_2\,dx_3 = -\frac{2Y\bar{\delta}}{a} - \frac{4Y}{a}\,\lambda(\bar{\delta}) + 4Y\int\limits_{0}^{\theta(\bar{\delta})} \frac{-æ(v,\bar{\delta})}{a}\,dv, \tag{2.1}$$

we obtain the equation

$$\frac{1}{a}\iint\limits_{\bar{A}} d(\,\overline{\psi^{\mu}}\,)_{\bar{\delta},\bar{s}}\,dx_2\,dx_3 = \mathcal{F}_{\bar{\delta}}\,d\,\frac{\bar{\delta}}{2}. \tag{2.2}$$

3. Approximate Estimates of the Latent Free Energy. We consider the simplest approximate estimates of the latent free energy of the microinhomogeneous stress-strain field near the tip of an opening-mode crack under conditions of plane stress.

Approximating the free boundary by the circular arc of radius $\rho(\bar{\delta})$ and by the segments of a broken line (Fig. 4), we find

$$æ(v,\bar{\delta}) = \rho(\bar{\delta})\exp{(v)} - (\rho(\bar{\delta}) + a - \lambda(\bar{\delta})). \tag{3.1}$$

The angle between the normal to the free boundary and the x_3 axis at its point of intersection with the rigid-plastic boundary is found to be

$$\theta(\bar{\delta}) = \ln\Big(1 + (a - \lambda(\bar{\delta}))/\rho(\bar{\delta})\Big). \tag{3.2}$$

If the angle $\alpha(\bar{\delta})$ is the opening angle that determines the above-mentioned circular arc, we obtain

$$\cos{(\alpha(\bar{\delta})/2)} = 1 - \lambda(\bar{\delta})/\rho(\bar{\delta}). \tag{3.3}$$

The last supplementary equation (incompressibility condition) can be written in the form

$$2\bar{\delta}a = \rho^2(\bar{\delta})(\alpha(\bar{\delta}) - \sin\alpha(\bar{\delta})). \tag{3.4}$$

By virtue of (3.1) and (3.2), we find

$$4Y\int\limits_{0}^{\theta(\bar{\delta})} \frac{-æ(v,\bar{\delta})}{a}\,dv = \frac{4Y}{a}\,\rho(\bar{\delta})\Big\{1 + \Big(1 + \frac{a-\lambda(\bar{\delta})}{\rho(\bar{\delta})}\Big)\Big[\ln\Big(1 + \frac{a-\lambda(\bar{\delta})}{\rho(\bar{\delta})}\Big) - 1\Big]\Big\}. \tag{3.5}$$

We consider the following asymptotic representation for the angle $\alpha(\bar{\delta})$:

$$\alpha(\bar{\delta}) = (\bar{\delta}/\Gamma)^{\gamma}, \qquad \bar{\delta} \to 0 \tag{3.6}$$

(Γ and γ are certain constants). This representation can be introduced into Eq. (3.4) to give

$$\rho(\bar{\delta}) = 2\sqrt{3a}\,\Gamma^{3\gamma/2}\,\bar{\delta}^{(1-3\gamma)/2}. \tag{3.7}$$

We assume that $1/3 \leqslant \gamma \leqslant 1$, i.e., the free boundary becomes straight as $\bar{\delta} \to 0$. It can also be shown [see (3.3), (3.6), and(3.7)] that

$$\lambda(\bar{\delta}) = (\sqrt{3a}/4)\,\Gamma^{-\gamma/2}\,\bar{\delta}^{(1+\gamma)/2}. \tag{3.8}$$

With allowance for (3.5), (3.7), and (3.8), the generalized force (2.1) takes the form

$$\mathcal{F}_{\bar{\delta}} = -\frac{2Y\bar{\delta}}{a} - \frac{Y\sqrt{3}}{\sqrt{a}}\,\Gamma^{-\gamma/2}\,\bar{\delta}^{(1+\gamma)/2} + \frac{2Y\sqrt{a}}{\sqrt{3}}\,\Gamma^{-3\gamma/2}\,\bar{\delta}^{(3\gamma-1)/2}.$$

If $1/3 \leqslant \gamma \leqslant 1$, the third term in the last asymptotic representation becomes the leading term: $\mathcal{F}_{\bar{\delta}} = (2Y\sqrt{a}/\sqrt{3})\Gamma^{-3\gamma/2}\,\bar{\delta}^{(3\gamma-1)/2}$.

For the free-boundary approximation considered above, the slip field can be constructed only if the condition $a - \lambda(\bar{\delta}) \leqslant \rho(\bar{\delta})[\exp(\pi/2) - 1]$ holds. Therefore, with allowance for (3.7), we obtain

$$\frac{1}{\Gamma^{3\gamma/2}} \leqslant \frac{2\sqrt{3}[\exp(\pi/2) - 1]}{\sqrt{a}\bar{\delta}^{(3\gamma-1)/2}}.$$

Assuming that the magnitude of crack opening is much smaller than the plate thickness, we can choose the constant Γ such that

$$\frac{1}{\Gamma^{3\gamma/2}} \leqslant \frac{2\sqrt{3}[\exp(\pi/2) - 1]}{a^{3\gamma/2}}.$$

In this case, the previous inequality is satisfied automatically. Then

$$\mathcal{F}_{\bar{\delta}} = 4Y\Big[\exp\Big(\frac{\pi}{2}\Big) - 1\Big]\Big(\frac{\bar{\delta}}{a}\Big)^{(3\gamma-1)/2}.$$

Using (2.2), we find

$$\iint\limits_{\bar{A}} (\,\overline{\psi^{\mu}}\,)_{\hat{\omega},\bar{s}}\,dx_2\,dx_3 = \frac{4Ya^2[\exp(\pi/2) - 1]}{3\gamma + 1}\Big(\frac{\bar{\delta}}{a}\Big)^{(3\gamma+1)/2}.$$

The right side of the last equation reaches the maximum for $\gamma = 1/3$. Since $\bar{\delta} \leqslant \hat{\omega}$, we obtain

$$\iint\limits_{\bar{A}} (\,\overline{\psi^{\mu}}\,)_{\hat{\omega},\bar{s}}\,dx_2\,dx_3 = 2Ya\Big[\exp\Big(\frac{\pi}{2}\Big) - 1\Big]\hat{\omega}. \tag{3.9}$$

Using Dugdale's formula

$$\hat{\omega} = \frac{8Yl}{\pi E}\ln(\sec\hat{\beta}), \qquad \hat{\beta} = \frac{\pi\hat{\sigma}_{22}^{\infty}}{2Y},$$

where E is Young's modulus, we finally obtain the rough estimate of the latent free energy (per unit length of the plastic zone)

$$\iint\limits_{\bar{A}} (\,\overline{\psi^{\mu}}\,)_{\hat{\omega},\bar{s}}\,dx_2\,dx_3 = \frac{16Y^2al}{\pi E}\Big[\exp\Big(\frac{\pi}{2}\Big) - 1\Big]\ln(\sec\hat{\beta}).$$

Refined estimates can be obtained if the more accurate approximations of the free surface of the localized plastic zone are used.

The effect of the latent free energy associated with the microinhomogeneity of plastic flow on the fracture of solids has been discussed in many publications. The results are summarized in [6]. When released, for example, in the cyclic loading of the crack, the latent free energy is spent for microfracturing of the elements located at the crack tip. In the case of hydrogen corrosion, the embrittlement of a metal near the crack tip accompanied by degradation of its plastic properties also leads to cracking, which, from the thermodynamic viewpoint, is the equalization of the microinhomogeneity of the stress-strain state of the localized plastic zone. In any case, the above-mentioned microfracture can be characterized by a scalar [7] or tensor [8] variable, which is usually called damage.

The simplest scheme of investigation implies a certain estimate of the total free surface of microdefects Σ. Assuming that the accumulated latent energy is spent for the nucleation and development of microdefects, we find

$$\frac{1}{2a} \iint\limits_{\bar{A}} (\,\overline{\psi^{\mu}}\,)_{\hat{\omega},\bar{s}}\, dx_2\, dx_3 = \frac{k_f \Sigma}{2\hat{c}a}, \tag{3.10}$$

where k_f is the energy spent for the formation of the unit free surface (the Griffith constant). The ratio $\Sigma/(2\hat{c}a)$ can be used as the simplest scalar measure of damage D.

With allowance for (3.9) and (3.10), we estimate the damage

$$D = \frac{Y}{k_f} \Big[\exp\Big(\frac{\pi}{2}\Big) - 1 \Big] \hat{\omega}, \tag{3.11}$$

which is the upper estimate, i.e., the maximum possible damage.

Since, for most metals, the critical damage D_c at which uncontrollable cracking occurs satisfies the inequality $0.2 \leqslant D_c \leqslant 0.8$, the estimate (3.11) allows one to calculate the crack opening $\hat{\omega}_c$ which corresponds to the onset of uncontrollable cracking at the crack tip.

By virtue of (3.11), the critical crack opening $\hat{\omega}_c$ is calculated from the formula

$$\hat{\omega}_c = \frac{D_c k_f}{Y(\exp(\pi/2) - 1)}$$

and depends on the critical value of the scalar damage parameter D_c.

Consequently, the critical crack opening can be one order of magnitude smaller than that predicted by the traditional theories (see, e.g., [9]), which ignore the effect of the damage localized near the crack tip in the plastic region on the state of the crack.

It should also be noted that the above-considered estimate of the critical crack opening is obtained under the assumption that the latent free energy localized near the crack tip is released *completely*, which implies considerable embrittlement of the end zone (for example, owing to the inflow of hydrogen atoms) or its reverse deformation (for example, under cyclic loading). Otherwise, this estimate fails.

REFERENCES

1. D. S. Dugdale, “Yielding of steel sheets containing slits,” *J. Mech. Phys. Solids*, **8**, 100–104 (1960).
2. G. T. Hanh and A. R. Rosenfield, “Local yielding and extension of crack under plane stress,” *Acta Met.*, **13**, 292–306 (1969).
3. R. Hill, *The Mathematical Theory of Plasticity*, Clarendon Press, Oxford (1950).
4. Yu. V. Zaitsev, *The Fracture Mechanics for Constructors* [in Russian], Vysshaya Shkola, Moscow (1991), pp. 73–77.
5. Yu. N. Radaev, “The limit state of a neck of arbitrary configuration in a rigid-plastic body,” *Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela*, No. 6, 69–75 (1988).
6. V. V. Novozhilov and Yu. I. Kadashevich, *Microstresses in Structural Materials* [in Russian], Mashinostroenie, Leningrad (1990).
7. L. M. Kachanov, *Introduction to Continuum Damage Mechanics*, Martinus Nijhoff, Dordrecht–Boston–Lancaster (1986).
8. S. Murakami and Yu. N. Radaev, “Mathematical model of a three-dimensional anisotropic state of damage,” *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 4, 93–110 (1996).
9. D. Broek, *Elementary Engineering Fracture Mechanics*, Noordhoff, Groningen (1974).